

B.Sc. VI SEMESTER

Mathematics

PAPER – I

DIFFERENTIAL EQUATIONS

UNIT-III

Legendre Equation And Functions

Syllabus:

Unit – I

Solutions of Legendre's equations in series, Legendre's functions-First and Second kind, Rodrigue's formula, Orthogonal properties, Legendre's Polynomial, Recurrence formulae.

-10HRS

Lecture Notes by
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LEGENDRE EQUATIONS AND FUNCTIONS

Definition: The differential equation of the form $\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + n(n+1)y = 0$ or

$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$ ----(1) is called Legendre equation, where n is a

positive integer. We now solve eqn.(1) in a series of decreasing powers of x. Let the

series of solution of (1) be $y = \sum_{r=0}^{\infty} a_r x^{m-r}$ ---(2) where $a_0 \neq 0$. Differentiating eqn.(2)

$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m-r) x^{m-r-1}$ and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m-r)(m-r-1) x^{m-r-2}$ then putting the values

of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$(1-x^2) \sum_{r=0}^{\infty} a_r (m-r)(m-r-1) x^{m-r-2} - 2x \sum_{r=0}^{\infty} a_r (m-r) x^{m-r-1} + n(n+1) \sum_{r=0}^{\infty} a_r x^{m-r} = 0$$

$$\sum_{r=0}^{\infty} a_r (m-r)(m-r-1) x^{m-r-2} - \sum_{r=0}^{\infty} a_r \{(m-r)(m-r-1) + 2(m-r) - n(n+1)\} x^{m-r} = 0 --(3)$$

$$\begin{aligned} & \text{where } (m-r)(m-r-1) + 2(m-r) - n(n+1) = (m-r)^2 - (m-r) + 2(m-r) - n^2 - n \\ & = (m-r)^2 - n^2 + (m-r) - n = (m-r-n)(m-r+n) + (m-r-n) \\ & = (m-r-n)(m-r+n+1). \end{aligned}$$

Eqn.(3) becomes

$$\sum_{r=0}^{\infty} a_r (m-r)(m-r-1) x^{m-r-2} - \sum_{r=0}^{\infty} a_r (m-r-n)(m-r+n+1) x^{m-r} = 0 --(4)$$

Eqn.(4) is an identity. To get the indicial equation, we equate the coefficients of the highest power of x (i.e. x^m) to zero, we obtain $a_0(m-n)(m+n+1)=0$ (put $r=0$ in 2nd summation). Since $a_0 \neq 0, (m-n)(m+n+1)=0$ i.e. $m=n, m=-(n+1)$ --(5). They are unequal and differ by an integer. The next lower power of x is $m-1$ (i.e. x^{m-1}).

So we equate to zero, the coefficient of x^{m-1} in eqn.(4) and obtain (put $r=1$ in 2nd summation)

$$a_1(m-1-n)(m-1+n+1)=0 \text{ i.e. } a_1(m-n-1)(m+n)=0 \text{ which gives}$$

$a_1=0$ as $(m-n-1)(m+n) \neq 0$ from(5). Again to find a relation in successive coefficients of a_r etc, equating the coefficients of x^{m-r-2} (replace r by $r+2$)

$$\begin{aligned} a_r(m-r)(m-r-1)-a_{r+2}[(m-r-n-2)(m-r+n-1)] &= 0 \\ a_{r+2} = -\frac{(m-r)(m-r-1)}{(m-r+n-1)(m-r-n-2)} a_r &\quad \dots \quad (7) \end{aligned}$$

Now since $a_1=0 \Rightarrow a_3=a_5=a_7=\dots=0$

For the values given by (5), there arise the following two cases

Case-I: When $m=n$

$$\text{From (7)} \quad a_2 = -\frac{n(n-1)}{2(2n-1)} a_0 \text{ for } r=0,$$

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)} a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} a_0 \text{ for } r=2 \text{ and so on}$$

Hence the series (2) becomes $y = \sum_{r=0}^{\infty} a_r x^{n-r} = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots$

$$\text{i.e. } y = a_0 \left[x^n - \frac{n(n-1)}{2.(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} - \dots \right] \quad \dots \quad (8)$$

which is the solution of eqn.(1) [as $a_1 = 0 \Rightarrow a_3 = a_5 = a_7 = \dots = 0$].

Case-II: When $m= -(n+1)$

We have $a_{r+2} = -\frac{(m-r)(m-r-1)}{(m-r-n-2)(m-r+n-1)} a_r$ from(7)

so that $a_2 = -\frac{(n+1)(n+2)}{2(2n+3)} a_0$ for $r=0$,

$a_4 = -\frac{(n+3)(n+4)}{4(2n+5)} a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} a_0$ for $r=2$ and so on.

$$\begin{aligned} \text{Hence the series (2) in this case becomes } y &= \sum_{r=0}^{\infty} a_r x^{n-r} = a_0 x^{-n-1} + a_1 x^{-n-3} + a_2 x^{-n-5} + \dots \\ &= a_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} x^{-n-5} + \dots \right] \quad \dots (9) \end{aligned}$$

This gives another solution of (1) in a series of descending power of x.

LEGENDRE'S FUNCTION OF FIRST KIND $P_n(x)$:

Definition: The Legendre equation is $(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \dots (1)$

The solution of the above equation in the series of descending powers of x is

$$y = a_0 \left[x^n - \frac{n(n-1)}{2.(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} - \dots \right] \text{ where } a_0 \neq 0 \text{ is an arbitrary}$$

constant. Now if n is a positive integer and $a_0 = \frac{1.3.5....(2n-1)}{n!}$ the above solution is

$$\text{called } P_n(x), \text{ so that } P_n(x) = \frac{1.3.5....(2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2.(2n-1)} x^{n-2} + \dots \right]$$

$P_n(x)$ is called the Legendre's function of the first kind.

Note-1: This is a terminating series. When n is even, it contains $\frac{n}{2} + 1$ terms, the last term

being $(-1)^{\frac{n}{2}} \cdot \frac{n(n-1)(n-2)(n-3)\dots2.1}{(2n-1)(2n-3)\dots(n+1).2.4.6\dots.n}$. And when n is odd it contains $\frac{1}{2}(n+1)$

terms and the last term in this case is $(-1)^{\frac{1}{2}(n-1)} \cdot \frac{n(n-1)(n-2)(n-3)\dots3.2}{(2n-1)(2n-3)\dots(n+1).2.4.6\dots.(n-1)} \cdot x$

Note : $P_n(x)$ is the solution of Legendre's equation (1) which is equal to unity when $x = 1$.

LEGENDRE'S FUNCTION OF THE SECOND KIND $Q_n(x)$:

Definition: Another solution of Legendre equation

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \quad (1), \text{ when } n \text{ is a positive integer, is}$$

$$y = a_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \dots \right] \text{ where } a_0 \neq 0 \text{ is an arbitrary constant.}$$

Now if we take $a_0 = \frac{n!}{1.2.3\dots(2n+1)}$ the above solution is called $Q_n(x)$, so that

$$Q_n(x) = \frac{n!}{1.2.3\dots(2n+1)} \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \dots \right]$$

$Q_n(x)$ is called the Legendre's function of the **second kind**.

The series for $Q_n(x)$ is a non-terminating series

GENERAL SOLUTION OF LEGENDRE'S EQUATION:

Since $P_n(x)$ and $Q_n(x)$ are two independent solutions of Legendre's equation, the most general solution of Legendre's equation is $y = AP_n(x) + BQ_n(x)$ where A and B are two arbitrary constants.

RODRIGUE'S FORMULA: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$

Proof: Let $v = (x^2 - 1)^n$ --(1), then $\frac{dv}{dx} = n(x^2 - 1)^{n-1} \cdot 2x$

Multiplying both sides by $(x^2 - 1)$,

$$\text{we get } (x^2 - 1) \frac{dv}{dx} = 2n(x^2 - 1)^n x \quad \text{or} \quad (x^2 - 1) \frac{dv}{dx} = 2nvx \text{ --- (2).}$$

Now differentiating (2), $(n+1)$ times by Leibnitz's theorem we have

$$(x^2 - 1) \frac{d^{n+2}v}{dx^{n+2}} + {}^{n+1} C_1 (2x) \frac{d^{n+1}v}{dx^{n+1}} + {}^{n+1} C_2 (2) \frac{d^n v}{dx^n} = 2n \left[x \frac{d^{n+1}v}{dx^{n+1}} + {}^{n+1} C_1 (1) \frac{d^n v}{dx^n} \right]$$

$$\text{If } y = v_n = \frac{d^n v}{dx^n}, \text{ then } (x^2 - 1)v_{n+2} + {}^{n+1} C_1 (2x)v_{n+1} + {}^{n+1} C_2 (2)v_n = 2n \left[xv_{n+1} + {}^{n+1} C_1 (1)v_n \right]$$

$$\text{or } (x^2 - 1)v_{n+2} + 2x[n^{n+1}C_1 - n]v_{n+1} + 2[n^{n+1}C_2 - n \cdot n^{n+1}C_1]v_n = 0$$

$$\text{Here } n^{n+1}C_1 - n = n+1 - n = 1, \text{ & } n^{n+1}C_2 - n \cdot n^{n+1}C_1 = \frac{(n+1)n}{2} - n(n+1) = -\frac{(n+1)n}{2}$$

$$\text{or } (x^2 - 1)v_{n+2} + 2xv_{n+1} - n(n+1)v_n = 0 \quad \dots \quad (3)$$

Eqn.(3) becomes $(x^2 - 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0$ as $y = \frac{d^n v}{dx^n}$

$$\text{or } (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{or} \quad (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

This shows that $y = \frac{d^n v}{dx^n}$ is a solution of Legendre's equation. $C \frac{d^n v}{dx^n} = P_n(x) \quad \dots \quad (4)$

where C is a constant.

But $v = (x^2 - 1)^n = (x+1)^n (x-1)^n$ so that

$$\frac{d^n v}{dx^n} = (x+1)^n \frac{d^n}{dx^n} (x-1)^n + n^{n+1}C_1 \cdot n(x+1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} (x-1)^n + \dots + (x-1)^n \frac{d^n}{dx^n} (x+1)^n = 0$$

when $x=1$, $\frac{d^n v}{dx^n} = 2^n \cdot n!$. All the other terms disappear as $(x-1)$ is a factor in every term

except first. Therefore when $x=1$, Eqn.(4) gives $C \cdot 2^n \cdot n! = P_n(1) = 1 \therefore C = \frac{1}{2^n \cdot n!}$

Substituting the values of v from (1) in (4) we have

$$P_n(1) = \frac{1}{2^n \cdot n!} \frac{d^n v}{dx^n} \quad \text{or} \quad P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

LEGENDRE POLYNOMIALS:

We have by Rodrigue's formula $P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2 - 1)^n$

If $n=0$, then $P_0(x) = \frac{1}{2^0 \cdot 0!} = 1$,

$$\text{If } n=1, \text{ then } P_1(x) = \frac{1}{2^1 \cdot 1!} \frac{d}{dx} (x^2 - 1) = \frac{1}{2} (2x) = x$$

$$\text{If } n=2, \text{ then } P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2} (x^2 - 1)^2 = \frac{1}{8} \frac{d}{dx} 2(x^2 - 1) \cdot 2x = \frac{1}{2} (x^3 - x) = \frac{1}{2} (3x^2 - 1)$$

$$\text{Similarly } P_3(x) = \frac{1}{2} (5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3),$$

$$P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5) \text{ -----and so on}$$

Example 1: Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ in terms of Legendre's Polynomials

$$\text{Solution: Since } P_3(x) = \frac{1}{2} (5x^3 - 3x), \text{ or } \frac{5}{2} x^3 = P_3(x) + \frac{3}{2} x,$$

$$\text{or } \frac{8}{5} \left(\frac{5}{2} x^3 \right) = \frac{8}{5} P_3(x) + \frac{8}{5} \left(\frac{3}{2} x \right) \text{ or } 4x^3 = \frac{8}{5} P_3(x) + \frac{12}{5} x \quad (1)$$

$$\text{Again } P_2(x) = \frac{1}{2} (3x^2 - 1) \quad \text{or} \quad \frac{3}{2} x^2 = P_2(x) + \frac{1}{2} \quad \text{or} \quad 4 \left(\frac{3}{2} x^2 \right) = 4P_2(x) + 2$$

$$\text{or} \quad 6x^2 = 4P_2(x) + 2 \quad (2)$$

$$\text{Now } f(x) = 4x^3 + 6x^2 + 7x + 2 = \left(\frac{8}{5} P_3(x) + \frac{12}{5} x \right) + (4P_2(x) + 2) + 7x + 2 \text{ t}$$

$$= \frac{8}{5} P_3(x) + 4P_2(x) + \frac{47}{5} x + 4$$

$$\text{As } P_1(x) = x \text{ & } P_0(x) = 1 \therefore f(x) = \frac{8}{5} P_3(x) + 4P_2(x) + \frac{47}{5} P_1(x) + 4P_0(x) \text{ Ans}$$

Example 2: Show that $8P_4(x) + 20P_2(x) + 7P_0(x) = 35x^4$

Solution: We know that $P_0(x) = \frac{1}{2^0 \cdot 0!} = 1$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$,

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$$

$$\therefore LHS = 8P_4(x) + 20P_2(x) + 7P_0(x)$$

$$= [(35x^4 - 30x^2 + 3) + 10(3x^2 - 1) + 7] = 35x^4 = RHS$$

Example 3: Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Legendre's polynomial.

$$\text{Solution: } P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad \therefore x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x \quad \text{---(1)}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad \therefore x^2 = \frac{2}{3}P_2(x) + \frac{1}{3} \quad \text{---(2)}$$

$$\begin{aligned} f(x) &= x^3 - 5x^2 + x + 2 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) - 5\left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right) + P_1(x) + 2P_0(x) \\ &= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x) \end{aligned}$$

B.Sc MATHEMATICS LECTURER NOTES

Class: B.Sc 6th Semester

Subject: Mathematics

Paper-I: Differential Equations

Unit-2: Legendre's Equations and Functions

Contents:

Legendre's equation, Legendre's polynomial, General solution of Legendre's equation Recurrence Formulae, Laplace's First and second definite integral for $P_n(x)$, orthogonal formula, Orthogonal Properties of Legendre's polynomials.

Possible Marks Distribution:

2marks -2Que or 3Que

5marks- 1Que or 2Que

10marks (5+5) - 1Que

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3. Legendre Equation and Function

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Date / /

The D.E of the form,

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \text{ is called}$$

Legendre equation.

Solution of this Legendre equation is called Legendre function.

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Obtain the series solution of LDE,

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

$$\text{or } (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

- Given the Legendre equation of order n is of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \rightarrow (1)$$

$$\text{the co-efficient } y'' = (1-x^2) = P_0(x)$$

$$\text{and } P_0(x) \neq 0 \text{ at } x=0$$

\therefore we apply power series method to solve this example

divide by $(1-x^2)$

$$y'' - \frac{2xy'}{1-x^2} + \frac{n(n+1)}{1-x^2} y = 0$$

$$P(x) = -\frac{2x}{1-x^2} \quad \& \quad Q(x) = \frac{n(n+1)}{1-x^2}$$

$$\text{at } x=0$$

$$P(x) = 0 \quad \& \quad Q(x) = n(n+1)$$

\therefore Both $P(x)$ and $Q(x)$ are analytic at $x=0$

Hence $x=0$ is ordinary pt.

Let $y = \sum_{m=0}^{\infty} a_m x^m$ be the power series of D.E (1)

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots \rightarrow (2)$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \& \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

and denote $n(n+1)$ by K

putting y, y', y'' in eqn (1)

$$-\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2 \sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} k a_m x^m = 0$$

$$-\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - \sum_{m=1}^{\infty} 2 m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

put $m-2 = p$ so that $m = p+2$ to 1st term

$$-\sum_{p=0}^{\infty} (p+2)(p+1) a_p x^p - \sum_{m=2}^{\infty} m(m-1) a_m x^m - \sum_{m=1}^{\infty} 2 m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

Replace p by m

$$-\sum_{m=0}^{\infty} (m+2)(m+1) a_m x^m - \sum_{m=2}^{\infty} m(m-1) a_m x^m - \sum_{m=1}^{\infty} 2 m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

$$-2a_0 x^0 + 6a_1 x^1 + \sum_{n=2}^{\infty} (m+2)(m+1) a_m x^m - \sum_{m=2}^{\infty} m(m-1) a_m x^m$$

$$-2a_1 x^1 - \sum_{m=2}^{\infty} 2 m a_m x^m + k a_0 x^0 + k a_1 x^1 + \sum_{m=2}^{\infty} k a_m x^m = 0$$

$$-x^0 [2a_0 + ka_0] + x^1 [6a_1 - 2a_1 + ka_1] + \sum_{m=2}^{\infty} [(m+2)(m+1) a_m - m(m-1) a_m + k a_m] x^m = 0$$

$$-m(m+1) a_m - 2 m a_m + k a_m] x^m = 0$$

equating to zero of co-efficient of $x, x^0 \dots$

$$2a_0 + ka_0 = 0 \\ a_0 = -\frac{ka_0}{2} = -\frac{n(n+1)a_0}{2}$$

$$6a_1 - 2a_1 + ka_1 = 0 \\ a_1 = \frac{2a_1 - ka_1}{6} = \frac{(2-k)a_1}{6} = \frac{2-n(n+1)a_1}{6}$$

$$= \frac{2-n^2-n}{6} a_n = -\frac{[(n+2)(n-1)]}{6} a_n$$

$$\begin{aligned}
 -(m+1)(m+2)a_{m+2} &= m(m-1)a_m + 2ma_m - ka_m \\
 a_{m+2} &= a_m [m^2 - m + 2m - k] \\
 &= a_m [m^2 + m - k] \\
 a_{m+2} &= \frac{a_m [m^2 + m - n^2 - n]}{(m+1)(m+2)} \quad \forall m \geq 2
 \end{aligned}$$

$$\begin{aligned}
 \text{put } m=2, a_4 &= \frac{a_2 (6 - n^2 - n)}{12} \\
 &= -\frac{n(n+1)a_0}{2} \frac{(6 - n^2 - n)}{12} \\
 &= -\frac{n(n+1)a_0}{24} [-(n+3)(n-2)] \\
 &= n(n+1)(n-2)(n+3)a_0
 \end{aligned}$$

$$\begin{aligned}
 \text{put } m=3, a_5 &= \frac{a_3 (12 - n^2 - n)}{20} \\
 &= -\frac{(n^2 + n - 12)}{20} - \frac{(n-1)(n+2)a_1}{6} \\
 &= \frac{(n-1)(n+2)(n-3)(n+4)a_1}{120}
 \end{aligned}$$

and so on.

$$\begin{aligned}
 y &= a_0 + a_1 x - \frac{n(n+1)a_0 x^2}{2} - \frac{(n-1)(n-2)a_1 x^3}{6} + \\
 &\quad \frac{n(n+1)(n-2)(n+3)a_0 x^4}{24} + \frac{(n-1)(n+2)(n-3)(n+4)a_1 x^5}{120} + \dots
 \end{aligned}$$

$$y = a_0 \left[1 - \frac{n(n+1)}{2} x^2 + \frac{n(n+1)(n-2)(n+3)}{24} x^4 + \dots \right]$$

$$a_1 \left[x - \frac{(n-1)(n+2)x^3}{6} + \frac{(n-1)(n+2)(n-3)(n+4)x^5}{120} + \dots \right] \rightarrow (4)$$

Let $u(x)$ and $v(x)$ respectively represents the two infinite series in eqn (4) so that

$$y = a_0 u(x) + a_1 v(x)$$

This is the series solution of legendre differential equation.

Rodriguis Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

Proof : If $u = (x^2 - 1)^n$

$$\text{then } u_1 = \frac{du}{dx} = n (x^2 - 1)^{n-1} 2x$$

$$= 2nx (x^2 - 1)^{n-1}$$

$$u_1 = \frac{2nx (x^2 - 1)^n}{(x^2 - 1)}$$

$$(x^2 - 1) u_1 = 2nx (x^2 - 1)^n$$

$$= 2nu_1 x.$$

$$2nu_1 + (1 - x^2) u_1 = 0$$

diff $(n+1)$ times by using Leibnitz's theorem

$$- (uv)_n = uv_n + n u_1 v_{n-1} + \frac{n(n-1)}{2!} u_2 v_{n-2} + \dots + u_n v$$

$$- (1-x^2) u_{n+2} + (n+1)(-2x) u_{n+1} + \frac{n(n+1)(-2)}{2!} u_n +$$

$$2n [x u_{n+1} + (n+1) u_n] = 0$$

$$- (1-x^2) u_{n+2} - 2x u_{n+1} + n(n+1) u_n = 0$$

$$- (1-x^2) \frac{d^2}{dx^2} (u_n) - 2x \frac{d}{dx} (u_n) + n(n+1) u_n = 0$$

This is Legendre ODE and $c \cdot u_n$ is its solution.

But the sum solution or polynomial solution of this eqn is $P_n(x)$

$$P_n(x) = c \cdot u_n$$

$$i \cdot c \cdot P_n(x) = c \cdot \frac{d^n}{dx^n} (u)$$

$$= c \cdot \frac{d^n}{dx^n} [(x^2 - 1)^n] \rightarrow \textcircled{1}$$

put $x = 1$

$$P_n(1) = c \cdot \frac{d^n}{dx^n} [(x^2 - 1)^n]_{x=1}$$

$$= c \cdot \frac{d^n}{dx^n} \left[(x+1)^n (x-1)^n \right]_{x=1}$$

$$= c \left[n! (x+1)^n + \text{term containing } (x-1)^n \right]_{x=1}$$

$$= c [n! (2)^n + 0]$$

$$= \frac{1}{n! 2^n}$$

put $c = \frac{1}{2^n n!}$ in eqn (1)

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

Legendre Polynomial

It may be obvious that the polynomial $u(x)$ and $v(x)$ containing alternate powers of x and a general form of the polynomial that represent either of them in descending powers of x can be represented in the form of

$$y = f(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \dots + F(x) \rightarrow (1)$$

where $F(x) = \begin{cases} a_0 & \text{if } n \text{ is even} \\ a_1 & \text{if } n \text{ is odd} \end{cases}$

We note that a_n is the coefficient of x^n in the series solution of D.E and we have obtained.

$$a_{n+2} = -\frac{[n(n+1) - r(r+1)] a_r}{(r+1)(r+2)} a_n \rightarrow (2)$$

We can express a_{n-2}, a_{n-4}, \dots in terms of a_n .

Now replace r by $n-2$ in eqn (2)

$$a_n = -\frac{[n(n+1) - (n-2)(n-1)] a_{n-2}}{n(n-2)} a_n$$

$$= -\frac{[n^2 + n - (n^2 - 3n + 2)] a_{n-2}}{n(n-2)} a_n$$

$$= -\frac{(4n-2)}{n(n-2)} a_{n-2}$$

$$a_n = -\frac{2(2n-1)}{n(n-1)} a_{n-2}$$

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

replace a by $n-4$ in eqn (2)

$$a_{n-2} = -\frac{[n(n+1) - (n-4)(n-3)]}{(n-3)(n-2)} a_{n-4}$$

$$= -\frac{(n^2+n - (n^2-7n+12))}{(n-3)(n-2)} a_{n-4}$$

$$= -\frac{[8n-12]}{(n-3)(n-2)} a_{n-4}$$

$$= -\frac{4(2n-3)}{(n-3)(n-2)} a_{n-4}$$

$$a_{n-4} = -\frac{[(n-3)(n-2)]}{4(2n-3)} a_{n-2}$$

$$= -\frac{(n-3)(n-2)}{4(2n-3)} \times -\frac{n(n-1)}{2(2n-1)} a_n$$

$$= \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} a_n \text{ and so on}$$

using these values in eqn (1) we have

$$\begin{aligned} y = f(x) &= a_n x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} a_n + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} a_n x^{n-4} \\ &\quad + \dots + F(x) \\ &= a_n \left[x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)x^{n-4}}{2 \cdot 4 (2n-1)(2n-3)} \dots G(x) \right] \end{aligned}$$

where $G(x) = \begin{cases} \frac{a_0}{a_n} & \text{if } n \text{ is even} \\ \frac{a_1 x}{a_n} & \text{if } n \text{ is odd} \end{cases}$

If the constant a_n is to be chosen such that $y = f(x)$ becomes 1, when $x = 1$.

The polynomial so obtained called Legendre polynomial denoted by $P_n(x)$.

let us consider / choose

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \text{ to meet the above}$$

said requirement

i.e.

$$P_n(x) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-4} + \cdots + l(x) \right]$$

we obtained few Legendre polynomials by putting $n=0, 1, 2, 3$ and 4

$$P_0(x) = \frac{1}{1}(1) = 1$$

$$\text{Put } n=1, P_1(x) = \frac{1}{1!}(x) = x$$

$$\text{Put } n=2, P_2(x) = \frac{1 \cdot 3}{2!} \left[x^2 - \frac{2 \cdot 1}{2 \cdot 3} x^0 \right] = \frac{3}{2!} \left[x^2 - \frac{x^0}{3} \right]$$
$$= \frac{1}{2} (3x^2 - 1)$$

$$\text{Put } n=3, P_3(x) = \frac{1 \cdot 3 \cdot 5}{3!} \left[x^3 - \frac{3 \cdot 2 \cdot 1}{2 \cdot 5} x \right]$$
$$= \frac{5}{2} \left[x^3 - \frac{3x}{5} \right]$$
$$= \frac{1}{2} (5x^3 - 3x)$$

$$\text{Put } n=4, P_4(x) = \frac{1 \cdot 3 \cdot 5 \cdot 7}{4!} \left[x^4 - \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 7} + \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 4 \cdot 7 \cdot 5} \right]$$
$$= \frac{35}{8} \left[x^4 - \frac{6x^2}{7} + \frac{3}{35} \right]$$
$$= \frac{35}{8} \left[\frac{35x^4 - 30x^2 + 3}{35} \right]$$
$$= \frac{1}{8} [35x^4 - 30x^2 + 3]$$

It can be easily seen that all those expression gives 1 at $x=1$ in accordance with the definition of Legendre polynomial.

Legendre polynomial of first kind or Legendre function of degree n .

It is denoted and defined by

$$P_n(x) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \cdots \right]$$

Legendre function of second kind or Legendre polynomial of degree n

It is denoted and defined by

$$Q_n(x) = \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-(n+3)} + \right.$$

$$\left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 (2n+3)(2n+5)} x^{-(n+5)} + \cdots \right]$$

1 using R.F obtain the expression for $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$. Hence express x^2 , x^3 , x^4 in terms of legendre polynomials.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

$$\text{put } n=0, P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} [(x^2 - 1)^0] \\ = 1 \frac{d^0}{dx^0}(1) = 1$$

$$\text{put } n=1, P_1(x) = \frac{1}{2^1 1!} \frac{d^1}{dx^1} (x^2 - 1)^1 \\ = \frac{1}{2} (2x) = x.$$

$$\text{Put } n=2, P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2 - 1)^2 \\ = \frac{1}{8} \frac{d^2}{dx^2} [x^4 + 1 - 2x^2] \\ = \frac{1}{8} \frac{d}{dx} [4x^3 - 4x] \\ = \frac{1}{8} [12x^2 - 4] \\ = \frac{4}{8} [3x^2 - 1] = \frac{1}{2} (3x^2 - 1)$$

$$\text{Put } n=3, P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2 - 1)^3 \\ = \frac{1}{48} \frac{d^3}{dx^3} (x^4 + 1 - 2x^2)(x^2 - 1) \\ = \frac{1}{48} \frac{d^3}{dx^3} [x^6 - x^4 + x^2 - 1 - 2x^4 + 2x^2] \\ = \frac{1}{48} \frac{d^3}{dx^3} [x^6 - 3x^4 + 3x^2 - 1]$$

Formula:	$\frac{d^n}{dx^n} x^m = \frac{m!}{(m-n)!} x^{m-n}$	if $m \geq n$
	= 0	if $m < n$

$$P_3(x) = \frac{1}{48} \left[\frac{6!}{3!} x^3 - 3 \cdot \frac{4!}{1!} x + 0 - 0 \right]$$

$$P_3(x) = \frac{1}{48} (120x^3 - 72x)$$

$$= \frac{1}{2} [5x^3 - 3x]$$

$$\text{Put } n=4, P_4(x) = \frac{1}{24 \cdot 4!} \frac{d^4}{dx^4} (x^2 - 1)^4$$

$$= \frac{1}{384} \frac{d^4}{dx^4} [(x^6 - 3x^4 + 3x^2 - 1)(x^2 - 1)]$$

$$= \frac{1}{384} \frac{d^4}{dx^4} [x^8 - x^6 - 3x^6 + 3x^4 + 3x^4 - 3x^2 - x^2 + 1]$$

$$= \frac{1}{384} \frac{d^4}{dx^4} [x^8 - 4x^6 + 6x^4 - 4x^2 + 1]$$

$$= \frac{1}{384} \left[\frac{8!}{4!} x^4 - \frac{4 \cdot 6! x^2}{2!} + \frac{6 \cdot 4! x^0}{0!} + 0 \right]$$

$$= \frac{1}{384} [1680x^4 - 1440x^2 + 144]$$

$$= \frac{48}{384} [35x^4 - 30x^2 + 3]$$

$$= \frac{1}{8} [35x^4 - 30x^2 + 3]$$

$$\text{Put } n=5, P_5(x) = \frac{1}{25 \cdot 5!} \frac{d^5}{dx^5} (x^2 - 1)^5$$

$$= \frac{1}{3840} \frac{d^5}{dx^5} [(x^8 - 4x^6 + 6x^4 - 4x^2 + 1)(x^2 - 1)]$$

$$= \frac{1}{3840} \frac{d^5}{dx^5} [x^{10} - x^8 - 4x^8 + 4x^6 + 6x^6 - 6x^4 - 4x^4 + 4x^2 + x^2 - 1]$$

$$= \frac{1}{3840} \frac{d^5}{dx^5} [x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1]$$

$$= \frac{1}{3840} \left[\frac{10!}{5!} x^5 - \frac{5 \cdot 8!}{3!} x^3 + \frac{10 \cdot 6!}{1!} x^1 + 0 \right]$$

$$= \frac{1}{3840} [30240x^5 - 33600x^3 + 7200x]$$

$$= \frac{480}{3840} [63x^5 - 70x^3 + 15x]$$

$$= \frac{1}{8} [63x^5 - 70x^3 + 15x]$$

we now express x^2, x^3, x^4, x^5 in terms of legendre polynomial.

$$\text{consider, } P_2(x) = \frac{1}{2} (3x^2 - 1)$$

$$2P_2(x) = 3x^2 - 1$$

$$3x^2 = 2P_2(x) + P_0(x)$$

$$x^2 = \frac{1}{3} [2P_2(x) + P_0(x)]$$

$$\text{consider, } P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

$$2P_3(x) = 5x^3 - 3x$$

$$5x^3 = 2P_3(x) + 3x$$

$$x^3 = \frac{1}{5} [2P_3(x) + 3P_1(x)]$$

$$P_4(x) = \frac{1}{8} [35x^4 - 30x^2 + 3]$$

$$8P_4(x) = 35x^4 - 30x^2 + 3$$

$$35x^4 = 8P_4(x) + 30x^2 + 3$$

$$= 8P_4(x) + 30 \left[\frac{1}{3} (2P_2(x) + P_0(x)) \right] - 3P_0(x)$$

$$= 8P_4(x) + 10P_0(x) + 20P_2(x) - 3P_0(x)$$

$$= 8P_4(x) + 20P_2(x) + 7P_0(x)$$

$$x^4 = \frac{1}{35} [8P_4(x) + 20P_2(x) + 7P_0(x)]$$

$$P_5(x) = \frac{1}{8} [63x^5 - 70x^3 + 15x]$$

$$8P_5(x) = \frac{1}{8} [63x^5 - 70x^3 + 15x]$$

$$63x^5 = 8P_5(x) + 70x^3 - 15x$$

$$= 8P_5(x) + 70 \left[\frac{1}{5} (2P_3(x) + 3P_1(x)) \right] - 15x$$

$$x^5 = \frac{1}{63} [8P_5(x) + 14 (2P_3(x) + 3P_1(x))] - 15P_1(x)$$

$$= \frac{1}{63} [8P_5(x) + 28P_3(x) + 42P_1(x) - 15P_1(x)]$$

$$= \frac{1}{63} [8P_5(x) + 28P_3(x) + 27P_1(x)]$$

2 Express $x^3 + 2x^2 - 4x + 5$ in terms of Legendre polynomial

- let $f(x) = x^3 + 2x^2 - 4x + 5 \rightarrow ①$

we know that, $P_0(x) = 1$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Now, $5 = 1 \cdot 5 = 5P_0(x)$

$$x = P_1(x)$$

Now, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$2P_2(x) = 3x^2 - 1$$

$$3x^2 = 2P_2(x) + 1$$

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

Now consider, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$2P_3(x) = 5x^3 - 3x$$

$$5x^3 = 2P_3(x) + 3x$$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

equation (1) becomes

$$f(x) = \frac{2}{3}P_3(x) + \frac{3}{5}P_1(x) + 2\left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right) - 4P_1(x) + 5P_0(x)$$

$$f(x) = \frac{2}{3}P_3(x) + \frac{4}{3}P_2(x) - \frac{17}{5}P_1(x) + \frac{17}{3}P_0(x)$$

3 Express $4x^3 - 2x^2 - 3x + 8$ in terms of Legendre polynomial

- let $f(x) = 4x^3 - 2x^2 - 3x + 8 \rightarrow ①$

we know that, $P_0(x) = 1$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

$$\text{Now } 8 = 8x^1 = 8P_0(x)$$

$$x = P_1(x)$$

$$\text{Now } P_2(x) = \frac{1}{2} (3x^2 - 1)$$

$$2P_2(x) = 2x^2 - 1$$

$$3x^2 = 2P_2(x) + 1$$

$$x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)$$

$$\text{Now consider, } P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

$$2P_3(x) = 5x^3 - 3x$$

$$5x^3 = 2P_3(x) + 3x$$

$$x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_0(x)$$

Equation (1) becomes

$$f(x) = 4 \left[\frac{2}{5} P_3(x) + \frac{3}{5} P_0(x) \right] - 2 \left[\frac{2}{3} P_2(x) + \frac{1}{3} P_0(x) \right]$$

$$= -3 [P_1(x)] + 8P_0(x)$$

$$f(x) = \frac{8}{5} P_3(x) + \frac{12}{5} P_1(x) - \frac{4}{3} P_2(x) - \frac{2}{3} P_0(x) - 3P_1(x) + 8P_0(x)$$

$$= \frac{8}{5} P_3(x) - \frac{4}{3} P_2(x) - \frac{3}{5} P_1(x) + \frac{22}{3} P_0(x)$$

5 Show that $x^4 = \frac{1}{35} [8P_4(x) + 20P_2(x) + 7P_0(x)]$

$$\text{Given, } x^4 = \frac{1}{35} [8P_4(x) + 20P_2(x) + 7P_0(x)]$$

$$\text{WKT, } P_0(x) = 1, P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$8P_4(x) = 18(35x^4 - 30x^2 + 3)$$

$$\begin{aligned} \text{Consider } \frac{1}{35} [8P_4(x) + 20P_2(x) + 7P_0(x)] &= \frac{1}{35} [8P_4(x) + 20 \cdot \frac{1}{2}(3x^2 - 1) + 7(1)] \\ &= \frac{1}{35} \left[8x^4 (35x^4 - 30x^2 + 3) + 10 \left(\frac{1}{2}(3x^2 - 1) + 7 \right) \right] \end{aligned}$$

$$= \frac{1}{35} [35x^4 - 30x^2 + 3 + 30x^2 - 10 + 7]$$

$$= \frac{1}{35} [35x^4 - 30x^2 + 30x^2 - 10 + 10] = \frac{1}{35} (35x^4) = x^4$$

$$\Rightarrow x^4 = \frac{1}{35} [8P_4(x) + 20P_2(x) + 7P_0(x)]$$

4 Express $4x^3 + 6x^2 + 7x + 2$ in terms of Legendre polynomial

$$- \quad \text{Let } f(x) = 4x^3 + 6x^2 + 7x + 2 \rightarrow (1)$$

$$\text{we get } P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\text{Now, } 2 = 2 \times 1 = 2P_0(x)$$

$$x = P_1(x)$$

$$\text{Now, } P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$3x^2 - 1 = 2P_2(x)$$

$$3x^2 = 2P_2(x) + 1$$

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$$\text{Now consider } P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$5x^3 - 3x = 2P_3(x)$$

$$5x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

Equation (1) becomes

$$f(x) = 4\left(\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right) + 6\left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right) + 7P_1(x)$$

$$+ 2P_0(x)$$

$$= \frac{8}{5}P_3(x) + \frac{12}{5}P_1(x) + 4P_2(x) + 2P_0(x) + 7P_1(x) + 2P_0(x)$$

$$= \frac{8}{5}P_3(x) + \frac{14}{5}P_1(x) + 4P_2(x) + 4P_0(x)$$

$$f(x) = \frac{8}{5}P_3(x) + 4P_2(x) + \frac{4}{5}P_1(x) + 4P_0(x)$$

5 Show that $\int_{-1}^1 P_n(x) dx = 0$, $n \neq 0$ and
 $\int_{-1}^1 P_n(x) dx = 2$, $n = 0$

- we know that $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$

Integrating we get

$$\begin{aligned} \int_{-1}^1 P_n(x) dx &= \frac{1}{2^n n!} \int_{-1}^1 \frac{d^n}{dx^n} (x^2 - 1)^n dx \\ &= \left[\frac{1}{2^n n!} \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \right]_{-1}^1 \\ &= \frac{1}{2^n n!} [0 - 0] = 0 \end{aligned}$$

$$\int_{-1}^1 P_0(x) dx = 0$$

when $n = 0$

$$\begin{aligned} \int_{-1}^1 P_0(x) dx &= \int_{-1}^1 I dx \quad \therefore P_0(x) = 1 \\ &= [x]_{-1}^1 \\ &= 1 + 1 = 2 \end{aligned}$$

Generating Function for Legendre polynomial

The function $(1 - 2xz + z^2)^{-1/2}$ is called a generating function for Legendre polynomial

Imp Show that $P_n(x)$ is the co-efficient of z^n in expression of $(1 - 2xz + z^2)^{-1/2}$ in ascending power of z .

OR

$$\text{Show that } (1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$$

$$\text{Proof: } (1 - 2xz + z^2)^{-1/2} = (1 - z(2x - z))^{-1/2}$$

$$(1 - 2xz + z^2)^{-1/2} = 1 + \frac{z}{2}(2x - z) + \frac{1 \cdot 3}{2 \cdot 4} z^2 (2x - z)^2 + \dots$$

$$\dots + \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{2 \cdot 4 \cdot 6 \dots 2n-2} z^{n-1} (2x - z)^{n-1} + \frac{1 \cdot 3 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} z^n (2x - z)^n$$

Now co-efficient of z^n is in

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} z^n (2x-z)^n = \frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{2 \cdot 4 \cdot 6 \cdots 2n} (2x)^n$$

$$\left[\because z^n (2x-z)^n = z^n [(2x)^n - nC_1 (2x)^{n-1} z^1 + nC_2 (2x)^{n-2} z^2 + \dots] \right. \\ \left. = (2x)^n z^n - nC_1 (2x)^{n-1} z^{n+1} + nC_2 (2x)^{n-2} z^{n+2} + \dots \right]$$

$$\begin{aligned} \therefore \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} z^n (2x-z)^n &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} z^n \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) 2^n x^n}{2^n (1 \cdot 2 \cdots n)} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) x^n}{n!} \end{aligned}$$

Now co-efficient of z^n in

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} z^{n-1} (2x-z)^{n-1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} z^{n-1} (2x-z)^{n-1}$$

$$\begin{aligned} \left[\because z^{n-1} (2x-z)^{n-1} &= z^{n-1} [(2x)^{n-1} - nC_1 (2x)^{n-2} z^1 + \frac{n-1}{2} nC_2 (2x)^{n-3} z^2 + \dots] \right. \\ &\quad \left. = -(n-1) (2x)^{n-2} z^n \right] \end{aligned}$$

$$\begin{aligned} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} z^{n-1} (2x-z)^{n-1} &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} [-(n-1)(2x)^{n-2}] \\ &= -\frac{(1 \cdot 3 \cdot 5 \cdots (2n-3)) (n-1) (2x)^{n-2}}{2^{n-1} (1 \cdot 2 \cdots (n-1))} \\ &= -\frac{[(1 \cdot 3 \cdot 5 \cdots (2n-3)) (n-1) x \frac{(2n-1) n}{2} 2^{n-2} x^{n-2}]}{2^{n-1} (1 \cdot 2 \cdots (n-1)) (2n-1) n} \\ &= -\frac{1 \cdot 3 \cdot 5 \cdots (2n-3) (2n-1) x n x 2^{n-2} x^{n-2} (n-1)}{2^{n-2} (1 \cdot 2 \cdot 3 \cdots (n-1) n) 2 (2n-1)} \\ &= -\frac{1 \cdot 3 \cdot 5 \cdots (2n-3) (2n-1) n (n-1) x^{n-2}}{n! 2 x (2n-1)} \end{aligned}$$

and so on

The co-efficient of z^n in expansion of

$$(1 - 2xz + z^2)^{-1/2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \left[x^n - \frac{n(n-1)x^{n-2} + \dots}{2(2n-1)} \right]$$

$$= P_n(x)$$

$$\text{Hence } P_n(x) = (1 - 2xz + z^2)^{-1/2}$$

1 Prove that $P_n(1) = 1$

- we know that $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) z^n$
put $x = 1$

$$(1 - 2z + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1) z^n$$

$$(1 - z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1) z^n$$

$$(1 - z^2)^{-1} = \sum_{n=0}^{\infty} P_n(1) z^n$$

$$1 + z + z^2 + \dots + z^n + \dots = \sum_{n=0}^{\infty} P_n(1) z^n$$

equating the co-efficient of z^n

$$P_n(1) = 1$$

2 Prove that $P_n(-x) = (-1)^n P_n(x)$ and deduce

$$P_n(-1) = (-1)^n$$

- we have, $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) z^n$

put $x = -z$

$$(1 + 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-x) z^n \rightarrow (1)$$

put $z = -z$

$$(1 + 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) (-z)^n \rightarrow (2)$$

From (1) & (2)

$$\sum_{n=0}^{\infty} P_n(-x) z^n = \sum_{n=0}^{\infty} P_n(x) (-z)^n$$

$$\sum_{n=0}^{\infty} P_n(-x) z^n = \sum_{n=0}^{\infty} P_n(x) (-1)^n z^n$$

equating the co-efficient of z^n

$$P_n(-x) = (-1)^n P_n(x)$$

Put $x = 1$

$$P_n(-1) = (-1)^n P_n(1)$$

$$= (-1)^n (1) \quad \because P_0(1) = 1$$

$$= (-1)^n$$

3 Prove that $P_n'(-1) = (-1)^{n-1} \frac{1}{2} n(n+1)$

- The legendre polynomial $P_n(x)$ satisfy the legendre equation,

$$(1 - x^2) y'' - 2x y' + n(n+1)y = 0$$

we have,

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$$

putting $x = -1$ and using $P_n(-1) = (-1)^n$

$$0 - 2(-1) P_n'(-1) + n(n+1) P_n(-1) = 0$$

$$2 P_n'(-1) = -n(n+1) (-1)^n$$

$$P_n'(-1) = \frac{-1}{2} n(n+1) (-1)^n$$

$$= \frac{n(n+1)(-1)^{n-1}}{2(-1)}$$

$$P_n'(-1) = \frac{n(n+1)(-1)^{n-1}}{2}$$

6 Show that $\frac{(1-z^2)}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) P_n(x) z^n$

- we know that, $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) z^n \rightarrow (1)$

differentiating eqn (1) w.r.t z

$$\frac{-1}{2} (1-2xz+z^2)^{-3/2} (-2x+2z) = \sum_{n=0}^{\infty} P_n(x) \cdot n z^{n-1}$$

$$(1-2xz+z^2)^{-3/2} (-x+z) = \sum_{n=0}^{\infty} P_n(x) \cdot n z^{n-1}$$

$$\frac{(x-z)}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} P_n(x) \cdot n z^{n-1} \rightarrow (2)$$

multiplying eqn (2) by $2z$ we get

$$\frac{2xz - 2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} 2n P_n(x) z \cdot z^{n-1}$$

$$= \sum_{n=0}^{\infty} 2n P_n(x) z^n \rightarrow (3)$$

Adding eqn (1) & (3)

$$\frac{1}{2} + \frac{3}{2} \quad (1-2xz+z^2)^{-1/2} + \frac{2xz - 2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} P_n(x) z^n + \sum_{n=0}^{\infty} 2n P_n(x) z^n$$

$$\frac{2}{2} = 1 \quad (1-2xz+z^2)^{-1/2} + \frac{2xz - 2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} P_n(x) z^n [1 + 2n]$$

$$\frac{1-2xz+z^2 + 2xz - 2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (1+2n) P_n(x) z^n$$

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (1+2n) P_n(x) z^n.$$

7 Prove that $\frac{(1+z)}{z(\sqrt{1-2xz+z^2})} = \sum_{n=0}^{\infty} (P_n + P_{n+1}) z^n$

- consider RHS,

$$= \sum_{n=0}^{\infty} (P_n + P_{n+1}) z^n$$

$$= \sum_{n=0}^{\infty} P_n z^n + \sum_{n=0}^{\infty} P_{n+1} z^{n+1}$$

$$= \sum_{n=0}^{\infty} P_n z^n + \frac{1}{z} \sum_{n=0}^{\infty} P_{n+1} z^{n+1} \times 8\% \text{ by } z \rightarrow 1$$

$$\text{But } \sum_{n=0}^{\infty} P_n z^n = P_0 + P_1 z + P_2 z^2 + P_3 z^3 + \dots$$

$$\sum_{n=0}^{\infty} P_{n+1} z^{n+1} = P_1 z + P_2 z^2 + P_3 z^3 + \dots$$

$$+ z - P_0 \Rightarrow = -P_0 + P_0 + P_1 z + P_2 z^2 + P_3 z^3 + \dots$$

$$= -P_0 + \sum_{n=0}^{\infty} P_n z^n$$

putting $\sum_{n=0}^{\infty} P_{n+1} z^{n+1} = -P_0 + \sum_{n=0}^{\infty} P_n z^n$ in eqn (1)

$$= \sum_{n=0}^{\infty} P_n z^n + \frac{1}{z} \left[-P_0 + \sum_{n=0}^{\infty} P_n z^n \right]$$

$$= \int_{z=0}^{\infty} P_n z^n - \frac{P_0}{z} \quad : P_0 = 1$$

$$= \left(1 + \frac{z}{z} \right) \sum_{n=0}^{\infty} P_n z^n - \frac{1}{z}$$

$$= \frac{(z+1)}{z} (1 - 2z z + z^2)^{-1/2} - \frac{1}{z}$$

$$= \frac{(z+1)}{z(1-2z+z^2)^{1/2}} - \frac{1}{z} = LHS.$$

8 Orthogonality properties of the Legendre polynomial

$$\int_{-1}^1 P_m(x) \cdot P_n(x) dx = \begin{cases} 0 & , \text{ if } m \neq n \\ \frac{2}{2n+1} & , \text{ if } m = n \end{cases}$$

$$\int_{-1}^1 P_m(x) \cdot P_n(x) dx = 2 \delta_{mn}$$

where δ_{mn} is the Kronecker delta

$$\text{defined by } \delta_{mn} = \begin{cases} 0 & , \text{ if } m \neq n \\ 1 & , \text{ if } m = n \end{cases}$$

OR

PROVE THAT $\int_{-1}^1 P_m(x) P_n(x) dx = 0$ if $m \neq n$

OR

PROVE THAT $\int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{2n+1}$ if $m = n$

PROOF: CASE 1: If $m \neq n$

The solution of Legendre eqn,

$$(1-x^2) u'' - 2x u' + n(n+1) u = 0$$

$$\text{and } (1-x^2) v'' - 2x v' + m(m+1) v = 0$$

are $P_m(x)$ and $P_n(x)$

$$\therefore (1-x^2) P_m''(x) - 2x P_m'(x) + m(m+1) P_m(x) = 0 \rightarrow (1)$$

$$\& (1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0 \rightarrow (2)$$

Multiplying eqn (1) by P_n & eqn (2) by P_m ,

$$(1-x^2) P_m'' P_n - 2x P_m' P_n + m(m+1) P_m P_n = 0$$

$$(1-x^2) P_n'' P_m - 2x P_n' P_m + n(n+1) P_n P_m = 0$$

$$- (1-x^2) [P_m'' P_n - P_n'' P_m] - 2x [P_m' P_n - P_n' P_m] + 0 = 0$$

$$[m(m+1) - n(n+1)] P_n P_m = 0$$

$$- (1-x^2) \frac{d}{dx} [P_n P_m' - P_n' P_m] - 2x [P_m' P_n - P_n' P_m] +$$

$$[m(m+1) - n(n+1)] P_n P_m = 0$$

$$- \frac{d}{dx} [(1-x^2) (P_n P_m' - P_m P_n')] = [n(n+1) - m(m+1)] P_n P_m$$

$$- \frac{d}{dx} [(1-x^2) (P_n P_m' - P_m P_n')] = [(n-m)(n+m-1)] P_n P_m$$

Integrating both sides w.r.t x from -1 to 1

we get

$$- \int_{-1}^1 P_n(x) P_m(x) dx \cdot (n-m)(n+m-1) = \int_{-1}^1 [(1-x^2) (P_n P_m' - P_m P_n')] dx$$

$$= (n-m)(n+m-1) \int_{-1}^1 P_n(x) P_m(x) dx = 0$$

Hence $n \neq m$

$$\int_{-1}^1 P_n(x) P_m(x) dx = 0$$

case II: when $m = n$

consider the generating function,

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$$

$$\& (1 - 2xz + z^2)^{-1/2} = \sum_{m=0}^{\infty} z^m P_m(x)$$

Multiplying the corresponding sides

$$(1 - 2xz + z^2)^{-1/2} (1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z^n z^m P_n(x) P_m(x)$$

$$(1 - 2xz + z^2)^{-1} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_m(x) P_n(x) \cdot z^{m+n}$$

Integrating both side w.r.t x

$$\int_{-1}^1 (1 - 2xz + z^2)^{-1} dx = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-1}^1 P_m(x) P_n(x) dx \cdot z^{m+n}$$

since $m = n$

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{-1}^1 [P_n(x)]^2 dx \cdot z^{2n} &= \int_{-1}^1 \frac{1}{1 - 2xz + z^2} dx \\ &= \left[\underbrace{\log(1 - 2xz + z^2)}_{(-2z)} \right]_{-1}^1 \\ &= \frac{-1}{2z} \left[\log(1 - 2z + z^2) - \log(1 + 2z + z^2) \right] \\ &= \frac{-1}{2z} \left[\log(1 - z)^2 - \log(1 + z)^2 \right] \\ &= \frac{-1}{2z} [2 \log(1 - z) - 2 \log(1 + z)] \\ &= \frac{-1}{z} [\log(1 - z) - \log(1 + z)] \\ &= \frac{1}{z} [\log(1 + z) - \log(1 - z)] \\ &= \frac{1}{z} \left[\left(z - \frac{z^2}{2} + \frac{z^3}{3} + \dots \right) - \left(-z + \frac{z^2}{2} - \frac{z^3}{3} + \dots \right) \right] \\ &= \frac{2}{z} \left[z + \frac{z^3}{3} + \frac{z^5}{5} + \dots \right] \\ &= \frac{2}{z} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)} \end{aligned}$$

$$\sum_{n=0}^{\infty} \int_{-1}^1 [P_n(x)]^2 dx = z^{2n} = \sum_{n=0}^{\infty} \frac{2z^{2n}}{2n+1}$$

now equating co-efficient z^{2n}

$$\int_{-1}^1 [P_n(x)]^2 dx = \frac{2}{2n+1}$$

Recurrence Formulae

- $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

- $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

- $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

- $P'_n(x) = xP'_{n-1}(x) + nP'_{n-1}(x)$

- $(x^2-1)P'_n(x) = n[xP_n(x) - P_{n-1}(x)]$

- $(x^2-1)P'_n(x) = (n+1)[P_{n+1}(x) - xP_n(x)]$

1. To show that $\frac{(n+1)P_n(x)}{n+1} = (2n+1)xP_n(x) = nP_{n-1}(x)$

- we know that,

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$$

diff w.r.t z

$$-\frac{1}{2} (1-2xz+z^2)^{-3/2} (-2x+2z) = \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$$

$$(1-2xz+z^2)^{-3/2} (x-z) = \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$$

multiplying both sides by $(1-2xz+z^2)$

$$(1-2xz+z^2)^{-1/2} (x-z) = \sum_{n=0}^{\infty} nz^{n-1} P_n(x) \cdot [(1-2xz+z^2)]$$

$$(x-z) \sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2) \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$$

$$\sum_{n=0}^{\infty} xz^n P_n(x) - \sum_{n=0}^{\infty} z^{n+1} P_n(x) = \sum_{n=0}^{\infty} nz^{n-1} P_n(x) - \sum_{n=0}^{\infty} 2xnz^n P_n(x) + \sum_{n=0}^{\infty} nz^{n+1} P_n(x)$$

$$\begin{aligned}
 - \sum_{n=0}^{\infty} x z^n P_n(x) &= [z P_0(x) + z^2 P_1(x) + z^3 P_2(x) + \dots + z^n P_n(x) \\
 &\quad + z^{n+1} P_{n+1}(x)] = [0 + z^0 P_1(x) + 2z^1 P_2(x) + 3z^2 P_3(x) + \dots \\
 &\quad \dots + (n+1) z^n P_{n+1}(x) + n z^{n-1} P_n(x)] - \sum_{n=0}^{\infty} 2n x z^n P_n(x) \\
 &\quad + [0 + z^2 P_1(x) + 2z^3 P_2(x) + 3z^4 P_3(x) + \dots \dots + \\
 &\quad \dots (n-1) z^n P_{n-1}(x) + n z^{n+1} P_{n+1}(x)]
 \end{aligned}$$

Now equating the coefficient of z^n on b.s

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2n x P_n(x) + (n-1) P_{n-1}(x)$$

$$x P_n(x) + 2n x P_n(x) = P_{n-1}(x) + (n-1) P_{n-1}(x) + (n+1) P_{n+1}(x)$$

$$x P_n(x) (2n+1) = P_{n-1}(x) [1+n-1] + (n+1) P_{n+1}(x)$$

$$x P_n(x) (2n+1) = P_{n-1}(x) (n) + (n+1) P_{n+1}(x)$$

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

2. Show that $n P_n(x) = x P'_n(x) - P'_{n-1}(x)$

- we know that

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \rightarrow (1)$$

diff eqn (1) w.r.t z

$$\frac{-1}{2} (1-2xz+z^2)^{-3/2} (-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

$$(1-2xz+z^2)^{-3/2} (x-z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x) \rightarrow (2)$$

diff eqn (1) w.r.t x .

$$\frac{-1}{2} (1-2xz+z^2)^{-3/2} (-2z) = \sum_{n=0}^{\infty} z^n P'_n(x)$$

$$z (1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} z^n P'_n(x) \rightarrow (3)$$

dividing eqn (2) by (3)

$$\frac{x-z}{z} = \frac{\sum_{n=0}^{\infty} n z^{n-1} P_n(x)}{\sum_{n=0}^{\infty} z^n P_n'(x)}$$

$$-(x-z) \sum_{n=0}^{\infty} z^n P_n'(x) = z \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

$$-\sum_{n=0}^{\infty} x z^n P_n'(x) - \sum_{n=0}^{\infty} z^{n+1} P_n'(x) = \sum_{n=0}^{\infty} n z^n P_n(x)$$

$$-\sum_{n=0}^{\infty} x z^n P_n'(x) - [z P_0'(x) + z^2 P_1'(x) + \dots + z^n P_n'(x) + z^{n+1} P_{n+1}'(x)]$$

$$= \sum_{n=0}^{\infty} n z^n P_n(x)$$

now equating the coefficient of z^n

$$- x P_n'(x) - z^0 P_{n-1}'(x) = n P_n(x)$$

$$- n P_n(x) = x P_n'(x) - z^0 P_{n-1}'(x)$$

Q. 3 show that $(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

(i) we know that,

$$(n+1) P_{n+1}'(x) = (2n+1)x P_n'(x) - n P_{n-1}'(x) \rightarrow (1)$$

now diff eqn (1) w.r.t x

$$(n+1) P_{n+1}'(x) = (2n+1) [x P_n'(x) + P_n(x)] - n P_{n-1}'(x) \rightarrow (2)$$

From Recurrence formula 2, we have

$$x P_n'(x) - P_{n-1}'(x) = n P_n(x)$$

$$x P_n'(x) = n P_n(x) + P_{n-1}'(x) \rightarrow (3)$$

Substituting eqn (3) in (2)

$$-(n+1) P_{n+1}'(x) = (2n+1) [n P_n(x) + P_{n-1}'(x) + P_n(x)] - n P_{n-1}'(x)$$

$$-(n+1) P_{n+1}'(x) = (2n+1) [(n+1) P_n(x) + P_{n-1}'(x)] - n P_{n-1}'(x)$$

$$-(n+1) P_{n+1}'(x) = (2n+1)(n+1) P_n(x) + (2n+1) P_{n-1}'(x) - n P_{n-1}'(x)$$

$$-(n+1) P_{n+1}'(x) = (2n+1)(n+1) P_n(x) + (n+1) P_{n-1}'(x)$$

$$P_{n+1}'(x) = (2n+1) P_n(x) + P_{n-1}'(x)$$

$$(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

213 4. show that $P_n'(x) = x P_{n-1}'(x) + n P_{n-1}(x)$

- we know that,

$$nP_n(x) = x P_n'(x) - P_{n-1}'(x) \quad (R.F-2)$$

$$(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \quad (R.F-3)$$

subtracting one from another

$$nP_n(x) - (2n+1) P_n(x) = [x P_n'(x) - P_{n-1}'(x)] - [P_{n+1}'(x) - P_{n-1}'(x)]$$

$$[n-2n-1] P_n(x) = x P_n'(x) - P_{n+1}'(x)$$

$$-(n+1) P_n(x) = x P_n'(x) - P_{n+1}'(x)$$

$$(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x)$$

Replace n by $n-1$

$$\frac{n}{n-1} P_n(x) = P_{n+1}'(x) - x P_{n-1}'(x)$$

$$P_n'(x) = x P_{n-1}'(x) + n P_{n-1}(x)$$

6 show that $(x^2 - 1) P_n'(x) = (n+1) [P_{n+1}(x) - x P_n(x)]$

- we know that,

$$(n+1) P_{n+1} = (2n+1) x P_n - n P_{n-1} \quad (R.F-1)$$

$$(n+1) P_{n+1} = (n+n+1) x P_n - n P_{n-1}$$

$$(n+1)[P_{n+1} - x P_n] = n x P_n - n P_{n-1} \rightarrow ①$$

But

$$(x^2 - 1) P_n'(x) = n [x P_n - P_{n-1}] \rightarrow ②$$

$$\Rightarrow (x^2 - 1) P_n'(x) = (n+1) [P_{n+1} - x P_n(x)]$$

7 show that $(x^2 - 1) P_n'(x) = n [x P_n(x) - P_{n-1}(x)]$

- we know that

$$x P_{n-1}'(x) + n P_{n-1}(x) = P_n'(x) \rightarrow ① \quad R.F-4$$

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x) \rightarrow ② \quad R.F-2.$$

multiple eqn ② by x.

$$x n P_n(x) = x^2 P_n'(x) - x P_{n-1}'(x) \rightarrow ③$$

subtract eqn ③ in ①

$$\Rightarrow x^2 P_n'(x) = x n P_n(x) + x P_{n-1}'(x)$$

i.e

$$x^2 P_n'(x) = x n P_n(x) + x P_{n-1}'(x)$$

$$P_n'(x) = \cancel{x P_{n-1}'(x)} + \underline{x^2 P_{n-1}'(x)}$$

$$(x^2 - 1) P_n'(x) = x n P_n(x) - n P_{n-1}'(x)$$

$$(x^2 - 1) P_n'(x) = n [x P_n(x) - P_{n-1}'(x)]$$

1 If $P_n(x)$ is the legendary polynomial of degree n and α is any number such that $P_n(\alpha) = 0$ show that $P_{n+1}(\alpha)$ and $P_{n+1}'(\alpha)$ are opposite sign.

- From the recurrence relation (1), we have

$$(2n+1)xP_n(x) - nP_{n-1}(x) = (n+1)P_{n+1}(x)$$

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

$$\text{put } x = \alpha.$$

$$(2n+1)\alpha P_n(\alpha) = (n+1)P_{n+1}(\alpha) + nP_{n-1}(\alpha)$$

$$0 = (n+1)P_{n+1}(\alpha) + nP_{n-1}(\alpha)$$

because $P_n(\alpha) = 0$.

$$(n+1)P_{n+1}(\alpha) = -n P_{n-1}(\alpha)$$

$$\frac{P_{n+1}(\alpha)}{P_{n-1}(\alpha)} = \frac{-n}{n+1} \rightarrow (3)$$

as n is a -ve integer so RHS of (3) is -ve

Hence $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ are opposite sign.

2 Prove that $\int P_n(x) dx = \frac{1}{2n+1} [P_{n+1}(x) - P_{n-1}(x)] + C$

- we know that

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

Integrating we get w.r.t x .

$$(2n+1) \int P_n(x) dx = P_{n+1}(x) - P_{n-1}(x) + C$$

$$\int P_n(x) dx = \frac{1}{2n+1} [P_{n+1}(x) - P_{n-1}(x)] + C.$$

3 Show that $P_{n+1}' + P_n' = P_0 + 3P_1 + 5P_2 + \dots + (2n+1)P_n$

- L.H.S R.F - (3)

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

$n = 0, 1, 2, \dots, n$ and add.

$$3P_1 = P_2' - P_0'$$

$$5P_2 = P_3' - P_1'$$

$$7P_3 = P_4' - P_2'$$

⋮

$$(2n-3)P_{n-3} = P_{n-1}' - P_{n-3}'$$

$$(2n-1)P_{n-1} = P'_n - P'_{n-1}$$

$$(2n+1)P_n = P'_{n+1} + P'_{n-1}$$

adding all we get

$$3P_1 + 5P_2 + \dots + (2n+1)P_n = P'_n + P'_{n+1} - P'_0 - P'_{-1}$$

$$= P'_n - P'_{n+1} - 0 - P'_0$$

since $P_0 = 1$ and $p = x$

$$P'_1 = 1 = P_0$$

$$P'_{n+1} + P'_{-n} = P_0 + 3P_1 + 5P_2 + \dots + (2n+1)P_n$$

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4 show that $\int_{-1}^1 (x^2 - 1) P_{n+1} P_n dx = \frac{2n(n+1)}{2n+3}$

- we have $f(x) = \sum_{m=0}^{\infty} c_m P_m(x)$

$$f(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + \dots + c_n P_n(x).$$

multiplying both sides by $P_m(x)$ we get

$$P_m(x) \cdot f(x) = c_0 P_0(x) P_m(x) + c_1 P_1(x) P_m(x) + \dots + c_m P_m^2(x).$$

$$\int_{-1}^1 f(x) P_m(x) dx = \int_{-1}^1 [c_0 P_0(x) P_m(x) + c_1 P_1(x) P_m(x) + \dots + c_m P_m^2(x) + \dots] dx.$$

$$= \left[0 + 0 + \dots + c_m \frac{2}{2m+1} + \dots \right]$$

$$= \frac{2c_m}{2m+1}$$

$$c_m = 2m+1 \int_{-1}^1 f(x) P_m(x) dx.$$

5 using the Rodrigues formula for $P_n(x)$ function

$$\int_{-1}^1 x^m P_n(x) dx = 0, m & n \text{ are +ve integers, } m < n$$

$$\int_{-1}^1 x^m P_n(x) dx = \int_{-1}^1 x^m \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

$$= \frac{1}{2^n n!} \int_{-1}^1 x^m \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

on integrating by parts we get

$$= \frac{1}{2^n n!} \left[x^m \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \right]_{-1}^1 - \int_{-1}^1 x^{m-1} m \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx$$

$$= 0 - \frac{m}{2^n n!} \int_{-1}^1 x^{m-1} \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx$$

similarly,

$$= \frac{m}{2^n n!} \int_{-1}^1 x^{m-1} \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx$$

$$= (-1)^2 \frac{m(m-1)}{2^n n!} \int_{-1}^1 x^{m-2} \frac{d^{n-2}}{dx^{n-2}} (x^2 - 1)^n dx$$

Integrating RHS $(m-2)$ times we get

$$\int_{-1}^1 x^m P_n(x) dx = (-1)^m \frac{m(m-1)}{2^n n!} \int_{-1}^1 \frac{d^{n-m}}{dx^{n-m}} (x^2 - 1)^n dx$$

$$= \frac{(-1)^m m!}{2^n n!} \int_{-1}^1 \frac{d^{n-m}}{dx^{n-m}} (x^2 - 1)^n dx$$

$$= \frac{(-1)^m m!}{2^n n!} \left[\frac{d^{n-m-1}}{dx^{n-m-1}} (x^2 - 1)^n \right]_{-1}^1$$

$$\int_{-1}^1 x^m P_n(x) dx = 0 //$$